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A STUDY ON THE STRUCTURE OF RIGHT TERNARY N-GROUPS

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ABSTRACT

Right ternary near-ring (RTNR) is a generalisation of its binary counterpart. In this paper right ternary N-group (or N-module) of an RTNR 'N' is defined and its basic algebraic properties are given. The substructures of a right ternary N-group (N-subgroups, normal subgroups, ideals) and the factor N-groups are also defined and homomorphism theorems on right ternary N-groups are obtained. The concept of faithful right ternary N-groups and monogenic right ternary N-groups are given in this generalised setting and every commutative RTNR is realized as M(Γ) (an RTNR of all mappings of an additive group Γ) where Γ is a faithful right ternary N-group. A simple monogenic right ternary N-group Γ is characterised in terms of maximal left ideals of N. If Γ is a faithful IFP right ternary N-group then N is shown as an IFP-RTNR. The three types of N-groups are defined and the relationships among them are established.

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KEYWORDS: Zero-Symmetric RTNR, Constant RTNR, Biunital Element, Normal Subgroup

1. INTRODUCTION

Near-rings are appropriate structures to study non-linear functions on finite groups. The set of all functions on groups under pointwise addition and composition are typical examples of near-rings. Just in the same way as R-modules over a ring R are used in ring theory, N-groups play an important role in the theory of near-rings.

The fundamental properties of algebraic structures can deeply be understood and further be developed in their *n*-ary context. Ternary algebraic structures [1, 3] have applications in Mathematical and theoretical physics. Lister [1] characterized additive subgroups of rings which are closed under triple ring product.

The authors introduced right ternary near-rings (RTNR) [7] and have studied their properties [4]. In this paper N-groups of an RTNR N are defined and their algebraic properties are studied. The substructures of N-group namely N-subgroups and ideals are considered. The factor right ternary N-groups are defined and homomorphism theorems on right ternary N-groups are obtained. It is proved that in a zero-symmetric RTNR every ideal of a right ternary N-group is an N-subgroup. The kernel of an N- homomorphism and the image of an onto N-homomorphism are shown as ideals. The definition of Noetherian quotient of two subsets of a right ternary N-group and the basic properties as given in [2] are established in this generalized setting.

The faithful right ternary N-groups and monogenic right ternary N-groups are defined and the process of embedding a commutative RTNR in M(Γ) (an RTNR of all mappings of Γ) where Γ is a faithful right ternary N-group is

described. If a right ternary N-group Γ is faithful IFP right ternary N-group then N is shown as an IFP-RTNR. The three types of right ternary N-groups are defined and the relationships among them are established.

2. PRELIMINARIES

In this section the basic definitions and results needed for the rest of the sections are given.

Definition 2.1 [7] Let N be a non-empty set together with a binary operation + and a ternary operation [ ]: $N \times N \times N \to N$. Then (N, +, [ ]) is a *right ternary near-ring* (RTNR) if

(RTNR-1) (N, +) is a group.

(RTNR-2) $[[xyz]\,uv] = [x\,[yzu]\,v] = [xy\,[zuv]] = [xyzuv]$ for every $x, y, z, u, v \in N$.

(RTNR-3) $[(x + y)\,z\,w] = [x\,z\,w] + [y\,z\,w]$ for every $x, y, z, w \in N$.

Similarly *left ternary near-ring* and *lateral ternary near ring* can be defined.

Example 2.2 [7] Let Γ be a group written additively with identity element 0. Then under point-wise addition and composition the following sets of mappings from Γ into Γ form RTNR.

- $M(\Gamma) = \{ f: \Gamma \to \Gamma \}$
- $M_0(\Gamma) = \{ f: \Gamma \to \Gamma \mid f(0) = 0 \}$
- $M_c(\Gamma) = \{ f: \Gamma \to \Gamma \mid f \text{ is constant} \}$.

Definition 2.3 [5] Let N be an RTNR. Then $N_0 = \{n \in N \mid [n\,0\,0] = 0\}$ is the *zero-symmetric part* of N .If $N = N_0$ then N is called *a zero-symmetric RTNR.*

Definition 2.4 [6] An element e in N is called an *idempotent* element if [eee] = e.

Definition 2.5 [7] Let N and N′ be RTNRs. Then a mapping $h: N \to N'$ is called an RTNR *homomorphism* if (i) h (m+ n) = h (m) + h (n) and (ii) h ([mnr]) = [h (m) h (n) h(r)] for every $m, n, r \in N$.

Definition 2.6 [7] Let N be a right ternary near-ring. Let I be a normal subgroup (N, +). Then I is called (i) *a right ideal* of N if $[\,INN\,] \subseteq I$ (ii) a *left ideal* if $[\,t\,t'\,(t'' + i)\,] - [\,t\,t'\,t''\,] \in I$ (iii) a *lateral ideal* if $[\,t\,(\,t' + i\,)\,t''\,] - [\,t\,t'\,t''\,] \in I$ for every $t, t', t'' \in N$, $i \in I$. I is called a *two-sided* ideal if it is a left and right ideal of N and I is an *ideal* of N if it is a left, right and lateral ideal of N.

Definition 2.7 [6] A non-empty subset H of N is called an *N- subgroup* of N if (i) H is a subgroup of (R, +) (ii) $[NNH] \subseteq H$ (iii) $[NHN] \subseteq H$ (iv) $[HNN] \subseteq H$.

If (i) and (ii) hold then H is called a *left* N-subgroup. If (i) and (iii) hold then H is called a *lateral* N-subgroup. If (i) and (iv) hold then H is called a *right* N-subgroup

Theorem 2.8 [6] If N is an RTNR and if $x \in N$, then [Nxx] is a left N-subgroup of N

Definition 2.9 [4] If N is an RTNR then $N_c = \{t \in N \mid [t\,0\,0] = t\}$ is called the *constant part* of N and N is a *constant RTNR* if $N = N_c$

Definition 2.10 [4] If N is an RTNR then an element $e \in N$ is a *left (resp. right, lateral) unital element* if $[e\ e\ x] = x$ ($[x\ e\ e] = x$, $[e\ x\ e] = x$) for every $x \in N$.

Theorem 2.11 [4] If I is an ideal an RTNR N then (N/I, +, [ ]) is an RTNR.

Definition 2.12 [4] If N is an RTNR then N is said to have *Insertion of Factor Property (IFP)* if for all $x, y, z \in N$ $[xyz] = 0 \Rightarrow [xuyvz] = 0$ for all $u, v \in N$.

3. RIGHT TERNARY N-GROUPS

In this section right ternary N-groups are defined and their structural properties are obtained. The substructures of N-group namely N-subgroups and ideals are considered. The factor N-groups are then defined and the most important theorem namely homomorphism theorem on right ternary N-groups is proved. It is proved that in a zero-symmetric RTNR every ideal of a right ternary N-group is an N-subgroup. The kernel of an N- homomorphism and the image of an onto N-homomorphism are shown as ideals.

Definition 3.1 Let (N, +, [ ]) be an RTNR and $(\Gamma, +)$ be a group with additive identity o. Then Γ is said to be a *right ternary N-group* if there exists a mapping $[\ ]_\Gamma : N \times N \times \Gamma \to \Gamma$ satisfying the conditions

(RTNG-1) $[n + m\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma + [m\ x\ \gamma]_\Gamma$

(RTNG-2) $[[n\ m\ u]\ x\ \gamma]_\Gamma = [n\ [m\ u\ x]\ \gamma]_\Gamma = [n\ m\ [u\ x\ \gamma]_\Gamma\ ]_\Gamma$ for all $\gamma \in \Gamma$ and $n, m, u \in N$.

Notation 3.2 A right ternary N-group Γ is denoted by ${}_N\Gamma$.

Example 3.3 (i) If $(\Gamma, +) = (N, +)$ then every RTNR is a right ternary N-group and is denoted by ${}_N N$

(ii) Let $(M(\Gamma), +, [\ ]) = N$ .Define $[\ ]_\Gamma : N \times N \times \Gamma \to \Gamma$ by $[f\ g\ \gamma]_\Gamma = f(g(\gamma))$, $f, g \in N$ and $\gamma \in \Gamma$.Then $(\Gamma, +, [\ ]_\Gamma)$ is a right ternary N-group.

In what follows 0 and o denote the identity elements of N and Γ respectively.

Proposition 3.4 Let ${}_N\Gamma$ be a right ternary N-group. Then

- $[0\ x\ \gamma]_\Gamma = o$ for every $x \in N$ and $\gamma \in \Gamma$.
- $[-n\ x\ \gamma]_\Gamma = -[n\ x\ \gamma]_\Gamma$ for every $\gamma \in \Gamma$ and $n, x \in N$.
- $[n\ u\ o]_\Gamma = o$ for every $n, u \in N_0$.
- $[n\ x\ \gamma]_\Gamma = [n\ 0\ o]_\Gamma$ for every $n \in N_c$, $x \in N$, $\gamma \in \Gamma$,

Proof: (i) Let $x \in N, \gamma \in \Gamma$.Consider $[0\ x\ \gamma]_\Gamma=[0+0\ x\ \gamma]_\Gamma = [0\ x\ \gamma]_\Gamma + [0\ x\ \gamma]_\Gamma$.Hence $[0\ x\ \gamma]_\Gamma = o$.

(ii) From (i), $o = [0\ x\ \gamma]_\Gamma = [n + (-n)\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma + [-n\ x\ \gamma]_\Gamma$.Hence $[-n\ x\ \gamma]_\Gamma = -[n\ x\ \gamma]_\Gamma$.

(iii) Let $n,u \in N_0$. Consider$[n\ u\ o]_\Gamma = [n\ u\ [0\ x\ \gamma]_\Gamma]_\Gamma = [[n\ u\ 0]\ x\ \gamma]_\Gamma = [0\ x\ \gamma]_\Gamma = o$.

(iv) $\forall\ n \in N_c$, $\gamma \in \Gamma$, $x \in N$ consider $[n\ x\ \gamma]_\Gamma = [[n\ 0\ 0]\ x\ \gamma]_\Gamma = [n\ [0\ 0\ x]\ \gamma]_\Gamma = [n\ 0\ [0\ x\ \gamma]_\Gamma]_\Gamma = [n\ 0\ o]_\Gamma$.

Definition 3.5 A subgroup Δ of ${}_N\Gamma$ is said to be an *N-subgroup* of ${}_N\Gamma$ if $[N\ N\ \Delta]_\Gamma \subseteq \Delta$

Remark 3.6 (i) A left N-subgroup of N is an N-subgroup of $_NN$.

(ii) If $_N\Gamma$ is described as an N-module then the N-subgroups are the sub modules of $_N\Gamma$.

Definition 3.7 If $_N\Gamma$ and $_N\Gamma'$ are any two right ternary N-groups then h : $_N\Gamma \to {}_N\Gamma'$ is an *N-homomorphism* if h $(\gamma + \delta) = h(\gamma) + h(\delta)\ \forall\ \gamma, \delta \in \Gamma$ and $h([n\ x\ \gamma]_\Gamma) = [n\ x\ h(\gamma)]_{\Gamma'}\ \forall n, x \in N$.

Note 3.8 (i) A one-one (resp. onto) N-homomorphism is called an *N-monomorphism (N-epimorphism)*

(ii) An N-homomorphism from Γ to Γ is called an *N-endomorphism*.

Notation 3.9 If there is a 1-1, onto, N-homomorphism from Γ to Γ' then it is denoted as $\Gamma \cong_N \Gamma'$ and is read as Γ is N-isomorphic to Γ'

Definition 3.10 If h: $\Gamma \to \Gamma'$ is an N-*homomorphism* then (i) ker h = $\{\gamma \in \Gamma \mid h(\gamma) = o'\}$.

(ii) Im h = $\{\gamma' \in \Gamma' \mid h(\gamma) = \gamma', \gamma \in \Gamma\}$.

Lemma 3.11 If h: $\Gamma \to \Gamma'$ is an N-homomorphism then (i) h (o) = o′ (ii) h $(-\gamma) = -$ h (γ).(iii) ker h, Im h are N_0-subgroups of Γ and Γ' respectively.

Proof The proof of (i) and (ii) are obvious.

(iii) Consider, $\gamma, \delta \in$ ker h. Then $h(\gamma - \delta) = h(\gamma) - h(\delta) = o'$. Hence $\gamma - \delta \in$ ker h.Also let for every n, u $\in$ N, $\gamma \in$ ker h,$[n\ u\ \gamma]_\Gamma \in [N_0\ N_0\ \ker h]_\Gamma$ .Consider $h([n\ u\ \gamma]_\Gamma) = [n\ u\ h(\gamma)]_\Gamma = [n\ u\ o']_{\Gamma'} = o' \Rightarrow [n\ u\ \gamma]_\Gamma \in \ker h\ \forall$ n, u $\in N_0$.Thus ker h is an N-subgroup of Γ. Obviously, Im h is subgroup of Γ'. Now, let $\gamma' \in$ Im h.Then $[n\ u\ \gamma']_{\Gamma'} = [n\ u\ h(\gamma)]_{\Gamma'} = h([n\ u\ \gamma]_\Gamma) \in$ Im h. Thus $[N\ N\ \text{Im}\ h]_{\Gamma'} \subseteq$ Im h.

Definition 3.12 (i) A subgroup Δ of Γ is called a *normal subgroup* of Γ if $\forall\ \gamma \in \Gamma, \delta \in \Delta, \gamma + \delta - \gamma \in \Delta$

(ii) A normal subgroup Δ of Γ is called an *ideal* of Γ if $\forall\ \gamma \in \Gamma$, $\forall\ \delta \in \Delta$ and $\forall$ n, x $\in$ N, $[n\ x\ (\gamma+\delta)]_\Gamma - [n\ x\ \gamma]_\Gamma \in \Delta$.

Remark 3.13 (i) {o} and Γ are trivial ideals of Γ.

(ii) If I is a left ideal of N then I is an ideal of the N-group $_NN$.

Lemma 3.14 Let Γ and Γ' be any two right ternary N-groups. Then

- If h: $\Gamma \to \Gamma'$ is an N-homomorphism then ker h is an ideal of Γ
- If h: $\Gamma \to \Gamma'$ is an onto N-homomorphism then Imh is an ideal of Γ'

Proof: Clearly ker h is a subgroup of Γ. Let $\gamma \in \Gamma$ and $\delta \in$ ker h. Consider $h(\gamma + \delta - \gamma) = h(\gamma) + h(\delta) - h(\gamma) = o'$ as $\delta \in$ ker h. Thus $\gamma + \delta - \gamma \in$ ker h and hence ker h is a normal subgroup of Γ.Let n, x $\in$ N, $\gamma \in \Gamma$, $\delta \in$ ker h. Consider $h([n\ x\ (\gamma+\delta)]_\Gamma - [n\ x\ \gamma]) = [n\ x\ h(\gamma+\delta)]_\Gamma - [n\ x\ h(\gamma)] = [n\ x\ h(\gamma)]_\Gamma - [n\ x\ h(\gamma)] =$ o.Hence $[n\ x\ (\gamma+\delta)]_\Gamma - [n\ x\ \gamma] \in$ ker h. Thus ker h is an ideal of Γ.

(ii) If $\eta', \delta' \in$ Imh then $\eta' - \delta' = h(\eta) - h(\delta) = h(\eta - \delta) \in$ Imh.Now let $\gamma' \in \Gamma'$ and $\delta' \in$ Imh.Then $\gamma' + \delta' - \gamma' = h(\gamma) + h(\delta) - h(\gamma) = h(\gamma + \delta - \gamma) \in$ Imh. Also if $\delta' \in$ Im h then $[n\ u\ \gamma'+\delta']_{\Gamma'} - [n\ u\ \gamma'] = [n\ u\ h(\gamma+\delta)]_{\Gamma'} - [n\ u\ h(\gamma)]_{\Gamma'} = h([n\ u\ \gamma + \delta]_\Gamma -$

$[n\ u\ \gamma]_\Gamma$)∈ Im h. Thus Im h is an ideal of Γ'.

Proposition 3.15: If N is a zero-symmetric RTNR and if Δ is an ideal of $_N\Gamma$ then Δ is an N-subgroup of $_N\Gamma$.

Proof: Let N be a zero-symmetric RTNR and Δ be an ideal of $_N\Gamma$.Then Δ is a subgroup of $_N\Gamma$. Now let n, x ∈ N and δ ∈ Δ.Consider $[n\ x\ \delta]_\Gamma = [n\ x\ \delta]_\Gamma - o = [n\ x\ \delta]_\Gamma - [n\ x\ o]_\Gamma = [n\ x\ o + \delta]_\Gamma - [n\ x\ o]_\Gamma \in \Delta$ as Δ is an ideal of $_N\Gamma$.Thus Δ is an N-subgroup of $_N\Gamma$.

Remark 3.16 Let Δ be an ideal of a right ternary N-group Γ. Define for a,b∈Γ, a ~ b iff a –b∈Δ.Then this relation is an equivalence relation on Γ and the corresponding equivalence class of a∈Γ is given by $[a] = a + \Delta = \{ a + \delta \mid \delta \in \Delta\}$.

Theorem 3.17 If Δ is an ideal of a right ternary N-group Γ and $\Gamma/\Delta = \{\gamma + \Delta \mid \gamma \in \Gamma\}$ then Γ/Δ is a right ternary N-group.

Proof: Let Δ be an ideal of a right ternary N-group Γ and $\Gamma/\Delta = \{\gamma + \Delta \mid \gamma \in \Gamma\}$.

Define for every γ,γ′ ∈ Γ, + and $[\ ]_{\Gamma/\Delta}$ as $(\gamma + \Delta) + (\gamma' + \Delta) = \gamma + \gamma' + \Delta$ and $[\ ]_{\Gamma/\Delta} : N \times N \times \Gamma/\Delta \to \Gamma/\Delta$ by $[n\ m\ \gamma + \Delta]_{\Gamma/\Delta} = [n\ m\ \gamma\ ]_\Gamma + \Delta$. Then (Γ/Δ,+) is a group and $[n + m\ x\ (\gamma + \Delta)]_{\Gamma/\Delta} = [\ n{+}m\ x\ \gamma\ ]_\Gamma + \Delta = [n\ x\ \gamma]_\Gamma + \Delta + [m\ x\ \gamma]_\Gamma + \Delta$. Similarly$[[n\ m\ u]\ x\ (\gamma + \Delta)]_{\Gamma/\Delta} = [[n\ m\ u\ ]x\ \gamma\ ]_\Gamma + \Delta = [\ n\ m\ [u\ x\ \gamma]_\Gamma + \Delta]_{\Gamma/\Delta} = [n\ m\ [u\ x\ \gamma + \Delta]_{\Gamma/\Delta}]_{\Gamma/\Delta}$.Thus Γ/Δ is a right ternary N-group called a *factor right ternary N-group* (or) *quotient right ternary N-group.*

Remark 3.18 If I is an ideal of a right ternary N- group N then N/I is an N-group.

Theorem 3.19 (Homomorphism theorem for right ternary N-groups)

- If Δ is an ideal of $_N\Gamma$ then the canonical mapping π:Γ→Γ/Δ is an N-epimorphism and Γ/Δ is an N-homomorphic image of Γ
- If h: Γ → Γ′ is an N-epimorphism with ker h = Δ then $\Gamma/\Delta \cong_N \Gamma'$.

Proof: (i) The natural projection from Γ to Γ/Δ is the N-epimorphism, π: Γ → Γ/Δ with π(γ) = γ + Δ for every γ ∈ Γ.Hence Γ/Δ is an N-homomorphic image of Γ.

(ii) Let h: Γ → Γ′ be an N-epimorphism with ker h = Δ. Then using the mapping θ: Γ/Δ→Γ' defined by θ (γ + Δ) = h(γ) for every γ ∈ Γ,it follows that $\Gamma/\Delta \cong_N \Gamma'$.

Theorem 3.20 Let h: Γ → Γ′ be an N-epimorphism. Then

- h induces a one-one correspondence between the set of all ideals of $_N\Gamma$ containing ker h and the set of all ideals of $_N\Gamma'$
- $\dfrac{\Gamma/\Delta}{\Delta'/\Delta} \cong_N \Gamma/\Delta'$, for all ideals Δ′ of Γ with Δ⊆ Δ'.

Proof: Let h: Γ → Γ′ be an N-epimorphism with ker h. Then it is easily seen that h (Δ) is an ideal of $_N\Gamma'$. If Δ is an ideal of $_N\Gamma$ and $h^{-1}(\Delta')$ is an ideal of $_N\Gamma$ whenever Δ' is an ideal of $_N\Gamma'$.

- Let $\Im$ be the set of all ideals of Γ containing ker h and $\Im'$ be the set of all ideals of ${}_N\Gamma'$. Define $\theta : \Im \rightarrow \Im'$ by $\theta(\Delta) = h(\Delta)$ for every $\Delta \in \Im$. Then θ sets up a 1-1 correspondence between $\Im$ and $\Im'$.

- For all ideals Δ' of Γ containing an ideal Δ define $\psi : \Gamma/\Delta \rightarrow \Gamma/\Delta'$ by $\psi(\gamma + \Delta) = \gamma + \Delta'$ for every $\gamma + \Delta \in \Gamma/\Delta$.Then ψ is an onto N- homomorphism with ker $\psi = \Delta'/\Delta$ and hence by Theorem 3.19(ii), $\dfrac{\Gamma/\Delta}{\Delta'/\Delta} \cong_N \Gamma/\Delta'$

Remark 3.21 If N is a zero-symmetric RTNR then the above theorem holds for N-subgroups also.

Lemma 3.22 Let N be an RTNR. Then (i) for $x \in N$ and $\gamma \in \Gamma$, $[N\ x\ \gamma]_\Gamma$ is an N-subgroup of Γ w.r.to x . (ii) $[N_c\ 0\ o]_\Gamma \subseteq \Delta$ if Δ is an N-subgroup of Γ.

Proof: (i) Let N be an RTNR and $x \in N$, $\gamma \in \Gamma$. Let $\delta, \delta' \in [N\ x\ \gamma]_\Gamma = E$ (say).Then $\delta - \delta' = [n\ x\ \gamma]_\Gamma - [n'\ x\ \gamma]_\Gamma = [n - n'\ x\ \gamma]_\Gamma \in E$.Thus E is a subgroup of Γ.Consider $\delta \in [N\ N\ E]_\Gamma$ then $\delta = [u\ v\ [n\ x\ \gamma]_\Gamma]_\Gamma = [[u\ v\ n]\ x\ \gamma]_\Gamma \in [N\ x\ \gamma]_\Gamma = E$. Thus $[N\ N\ E]_\Gamma \subseteq E$. Hence E is an N-subgroup of Γ. (ii) Let Δ be an N-subgroup of Γ. Then $[N\ N\ \Delta]_\Gamma \subseteq \Delta$.But $[N\ 0\ o]_\Gamma \subseteq [N\ N\ \Delta]_\Gamma \subseteq \Delta$.Hence $[N_c\ 0\ o]_\Gamma \subseteq \Delta$ as $N = N_0 + N_c$ and $[N_0\ 0\ o]_\Gamma = \{ o \}$.

Remark 3.23 $[N_c\ 0\ o]_\Gamma = [N\ 0\ o]_\Gamma$ is the smallest of all N-subgroups of Γ and is denoted as Ω.

Proposition 3.24 If $N = N_0$ then

- $\Omega = \{o\}$.
- For $\gamma \in \Gamma$, $x \in N$, $\Omega = [N_c\ x\ \gamma]_\Gamma$.
- $\Omega \cong_N {}_N N_c$.

Proof. (i) Follows from Remark3.23.

(ii) Follows from Proposition 3. 4 (iv)

(iii) Define $\phi : N_c \rightarrow [N_c\ 0\ 0]_N$ by $\phi(n_c) = [n_c\ 0\ 0]_N\ \forall\ n_c \in N_c$.

Then ϕ is easily seen to be well defined and 1-1. Also $\phi(n_c + n'_c) = \phi(n_c) + \phi(n'_c)$. Now $\phi([n_c\ n'_c\ n''_c]_N) = [n_c\ n'_c\ [n''_c\ 0\ 0]_N] = [n_c\ n'_c\ \phi(n''_c)]_N$.Thus ϕ is an N-homomorphism and is obviously onto. Hence $\Omega \cong_N N_c$.

Definition 3.25 (i) A right ternary N-group Γ is said to be *simple* if {o} and Γ are the only ideals of ${}_N\Gamma$.

(ii) A right ternary N-group Γ is said to be *N-simple* if Ω and Γ are the only N-subgroups of ${}_N\Gamma$.

Proposition 3.26 If a right ternary N-group Γ is simple then all N-homomorphic images of ${}_N\Gamma$ are N-isomorphic either to {o} or to ${}_N\Gamma$.

Proof: Let ${}_N\Gamma'$ be the N-homomorphic image of ${}_N\Gamma$. Then there is an N-epimorphism h: ${}_N\Gamma \rightarrow {}_N\Gamma'$ with ker h. Since ${}_N\Gamma$ is simple, ker h = {o} or Γ.By Theorem 3.19(ii) $\Gamma/\ker h \cong \Gamma'$. i.e., $\Gamma/\{o\} \cong_N \Gamma'$or $\Gamma/\Gamma \cong_N \Gamma'$. Hence $\Gamma' \cong_N \Gamma$ or $\Gamma' \cong_N \{o\}$

Proposition 3.27 If $N = N_0$ and if a right ternary N-group Γ is simple then Γ is N-simple.

Proof: If N = N_0 then by Proposition 3.15 the ideals of Γ are N-subgroups of Γ. Since Γ is N-simple Ω and Γ are its only N-subgroups. Also as N = N_0, Ω = {o}. Hence {o} and Γ are the only ideals of Γ and hence Γ is simple.

Theorem 3.28 A proper ideal Δ of a right ternary N-group Γ is maximal in Γ if Γ/Δ is simple.

Proof: Let Δ be a proper ideal of a right ternary N-group Γ which is maximal. If Δ′/Δ is an ideal of Γ/Δ where Δ′ is an ideal of Γ and $\Delta \subseteq \Delta'$ then since Δ is a maximal ideal of Γ it is seen that Δ′/Δ = Δ or Δ′/Δ = Γ/Δ.Thus Δ′/Δ is maximal in Γ/Δ. Hence Γ/Δ is simple. Conversely, let Γ/Δ be simple and Δ′ be any ideal of Γ such that $\Delta \subseteq \Delta'$. Then Δ′/Δ is an ideal of Γ/Δ. This implies that Δ′/Δ = Δ or Γ/Δ as Γ/Δ is simple. Thus Δ′ = Δ or Γ.Hence Δ′ is a maximal ideal of Γ.

Analogous to the Noetherian quotient of two subsets Δ_1, Δ_2 of a right ternary N-group Γ in binary near-ring the following definition is given.

Definition 3.29 If N is an RTNR and Δ_1, Δ_2 are any two non-empty subsets of a right ternary N-group Γ, then $(\Delta_1 : \Delta_2) = \{x \in N \mid [x\ n\ \delta_2]_\Gamma \in \Delta_1, \forall\ n \in N, \delta_2 \in \Delta_2\}$, $(o : \Delta) = \{x \in N \mid [x\ n\ \delta]_\Gamma = o\ \forall\ n \in N, \delta \in \Delta\}$ and $(o: \delta) = \{x \in N \mid [x\ n\ \delta]_\Gamma = o\ \forall\ n \in N\}$ (*annihilator of* $\delta \in \Delta$).

Theorem 3.30 If Δ_1 and Δ_2 are any two non-empty subsets of a right ternary N-group Γ then the following assertions hold:

- $(\Delta_1 : \Delta_2)$ is a subgroup of N-group N if Δ_1 is a subgroup of $_N\Gamma$.
- $(\Delta_1 : \Delta_2)$ is a normal subgroup of $_N N$ if Δ_1 is normal subgroup of $_N\Gamma$.
- $(\Delta_1 : \Delta_2)$ is an N-subgroup of $_N N$ if Δ_1 is an N-subgroup of $_N\Gamma$.
- $(\Delta_1 : \Delta_2)$ is an ideal in $_N N$ if Δ_1 is an ideal of $_N\Gamma$.

Proof: (i) Since for every $\delta_2 \in \Delta_2$ and $n \in N$, $[0\ n\ \delta_2]_\Gamma = o \in \Delta_1$, $(\Delta_1 : \Delta_2)$ is a non-empty subset of N. If h_1, $h_2 \in (\Delta_1 : \Delta_2)$ then as Δ_1 is a subgroup of $_N\Gamma$, $h_1 - h_2 = [x-y\ n\ \delta_2]_\Gamma \in \Delta_1$.Thus $(\Delta_1 : \Delta_2)$ is a subgroup of N.

(ii) If $h \in (\Delta_1 : \Delta_2)$, $u \in N$ then $[u+h-u\ n\ \delta_2]_\Gamma \in \Delta_1$, as Δ_1 is normal subgroup of $_N\Gamma$.

(iii) Let $(\Delta_1 : \Delta_2) = H$ and let $w \in [N\ N\ H]$.Consider for $u, v \in N$, $h \in H$

$[w\ n\ \delta_2]_\Gamma = [u\ v\ h]\ n\ \delta_2]_\Gamma = [u\ v\ [h\ n\ \delta_2]_\Gamma]_\Gamma \in [N\ N\ \Delta_1]_\Gamma \subseteq \Delta_1$ as Δ_1 is an N-subgroup of $_N\Gamma$.

(iv) If Δ_1 is an ideal of $_N\Gamma$ then it is a normal subgroup of $_N\Gamma$and hence by (ii) $(\Delta_1 : \Delta_2)$ is a normal subgroup of $_N N$. Now consider $h \in (\Delta_1 : \Delta_2)$.Then $[[n\ x\ n_1+h]_N - [n\ x\ n_1]_N\ u\ \delta_2]_N = [[n\ x\ n_1+h]_N\ u\ \delta_2]_\Gamma - [[n\ x\ n_1]\ u\ \delta_2]_\Gamma = [n\ x\ ([n_1\ u\ \delta_2]_\Gamma + [h\ u\ \delta_2]_\Gamma)]_\Gamma - [n\ x\ [n_1\ u\ \delta_2]_\Gamma]_\Gamma \in \Delta_1$ as Δ_1 is an ideal of $_N\Gamma$.Thus $(\Delta_1 : \Delta_2)$ is an ideal of $_N N$.

Corollary 3.31 (i) If $_N\Gamma$ is a right ternary N-group then $(o : \gamma)$ is a left ideal of N $\forall\ \gamma \in \Gamma$.

(ii) If Δ is an ideal of $_N\Gamma$ then (o : Δ) is a two sided ideal of N.

Proof: (a) Take $\Delta_1 = \{o\}$ and $\Delta_2 = \{\gamma\}$in Theorem 3.30 (iv).Then it follows that $(o : \gamma)$ is a left ideal of N.

(b) Take $\Delta_1 = \{o\}$ and $\Delta_2 = \Delta$ in Theorem 3.30(iv).Then it follows that (o : Δ) is a left ideal of N. Let (o : Δ) = H and $x \in H$. Now for $u, v \in N$ consider$[x\ u\ [v\ n\ \Delta]_\Gamma]_\Gamma \subseteq [x\ u\ \Delta]_\Gamma \subseteq \{o\} \Rightarrow [[x\ u\ v]\ n\ \Delta]_\Gamma \subset \{o\} \Rightarrow [x\ u\ v] \in H \Rightarrow [H\ N\ N] \subseteq$

H. Hence H is a right ideal of N. Thus (o : Δ) is a two sided ideal of N.

Proposition 3.32 If Δ is a subset of a right ternary N-group Γ, then $(o:\Delta)=\bigcap_{\delta\in\Delta}(o:\delta)$ and (o : Δ) is a left ideal of N.

Proof. Let $\delta \in \Delta$ and $x \in (o:\delta)$.Then for all $n \in N$, $[x\, n\, \delta]_\Gamma = o \Rightarrow [x\, n\, \Delta]_\Gamma = \{o\} \Rightarrow x \in (o:\Delta)$

Let $y \in (o:\Delta)$. Then $[y\, n\, \Delta]_\Gamma = \{o\} \Rightarrow [y\, n\, \delta]_\Gamma = o\ \forall\ \delta \in \Delta$.Hence $y \in (0:\delta) \Rightarrow y \in \bigcap_{\delta\in\Delta}(o:\delta)$ .Thus $(o:\Delta)=\bigcap_{\delta\in\Delta}(o:\delta)$ and obviously (o : Δ) is a left ideal of N.

Proposition 3.33 If Γ and Γ′ are any two right ternary N-groups and ${}_N\Gamma \cong {}_N\Gamma'$ then (o : Γ) = (o' : Γ′).

Proof: Let Γ and Γ′ be any two right ternary N-groups such that ${}_N\Gamma \cong {}_N\Gamma'$.Then there exists $h: \Gamma \to \Gamma'$, $h(\gamma) = \gamma'$ such that h is 1-1 and N-epimorphism. Now let $x \in (o:\Gamma) \Rightarrow [x\, n\, \gamma]_\Gamma = o \Rightarrow h([x\, n\, \gamma]_\Gamma) = h(o) \Rightarrow [x\, n\, h(\gamma)]_\Gamma = o' \Rightarrow [x\, n\, \gamma']_\Gamma = o' \Rightarrow x \in (o':\Gamma')$. Thus $(o:\Gamma) \subseteq (o':\Gamma')$. Retracing the steps $(o':\Gamma') \subseteq (o:\Gamma)$. Hence (o : Γ) = (o' : Γ′).

Theorem 3.34 If N is an RTNR and Γ is a right ternary N-group then for every $x \in N$ and $\gamma \in \Gamma$, $N/(o:\gamma) \cong_N [N\, x\, \gamma]_\Gamma$.

Proof: Define for $x \in N$ and $\gamma \in \Gamma$, $h: N \to [N\, x\, \gamma]_\Gamma$ by $h(n) = [n\, x\, \gamma]_\Gamma\ \forall\ n \in N$. Then h is obviously well defined and onto. It is easily seen that $h(n+n') = h(n) + h(n')$.Now $h([n\, n'\, n'']) = [n\, n'\, [n''\, x\, \gamma]_\Gamma]_\Gamma = [n\, n'\, h(n'')]_\Gamma$. Thus h is an N-homomorphism and ker h = $\{n \in N \mid [n\, x\, \gamma]_\Gamma = o\} = (o:\gamma)$. Thus by Theorem 3.19, $N/(o:\gamma) \cong_N [N\, x\, \gamma]_\Gamma$.

4. FAITHFUL RIGHT TERNARY N-GROUP, MONOGENIC RIGHT TERNARY N-GROUP AND IFP RIGHT TERNARY N-GROUP

In this section faithful and monogenic right ternary N-groups are defined. It is proved that every commutative RTNR can be embedded in M (Γ) where Γ is a faithful right ternary N-group. A simple right ternary N-group Γ is characterised in terms of maximal left ideals of right ternary N-group N. If a right ternary N-group Γ is faithful IFP right ternary N-group then N is shown as an IFP-RTNR.

Definition 4.1 A right ternary N-group Γ is called a *faithful right ternary* N-group if (o : Γ) = {0}.

Example 4.2 (i) Let N = {0, a, b, c, x, y} and + be defined as in Table 1 and for $x, y, z \in N$, let the ternary operation [ ] be defined by [xyz] = (x. y).z where is defined as in Table 2 then N is an RTNR. Let Γ = {0, x, y}. Then (0:Γ) = {0}

Table 1

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

Table 2

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	0	b	b
c	0	0	0	0	c	c
x	0	0	0	0	x	x
y	0	0	0	0	y	y

(ii) Let N = {0, a, b, c, x, y} and + be defined as in Table 3 and for x, y, z ∈ N, let the ternary operation [ ] be defined by [xyz] = (x. y).z where. is defined as in Table 4 then N is an RTNR. Let Γ= {0, x, y}.Then (0:Γ) ≠ {0}.

Table 3

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

Table 4

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	a	a	a	a	a	a
b	a	a	b	b	b	b
c	a	a	c	c	c	c
x	0	0	x	x	x	x
y	0	0	y	y	y	y

Definition 4.3 Let Γ be a right ternary N-group of an RTNR 'N'. Then for x ∈ N there exists γ∈ Γ, N is *monogenic* by γ *w.r.to x* if $[N\ x\ \gamma]_\Gamma = \Gamma$ and N is *monogenic by γ* if there exists γ∈ Γ and for every x ∈ N, $[N\ x\ \gamma]_\Gamma = \Gamma$.

Definition 4.4 A right ternary N-group Γ is *strongly monogenic* if Γ is monogenic and $[Nx\gamma]_\Gamma$ = {o} or Γ for every x ∈ N and γ ∈ Γ.

Remark 4.5 In general, {o} is not the smallest N-subgroup of a right ternary N-group but in a monogenic N_0-group,{o} is the smallest N_0- subgroup.

Example 4.6 (i) Let N = {0, x, y, z}. Define + as in Table 5 and the ternary operation [ ] on N by [xyz] = (x. y).z for every x, y, z in N where is defined as in Table 6. Then (N, +, [ ])is a right ternary near-ring. Let Γ = N. Then Γ is a right ternary N-group and is monogenic by γ = z w.r.to y as $[Ny\gamma]_\Gamma = [Nyz]_\Gamma = \Gamma$. Moreover Γ is strongly monogenic as $[Nx\gamma]_\Gamma$ = {0} or Γ for all x∈N.

Table 5

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table 6

.	0	x	y	z
0	0	0	0	0
x	0	0	x	x
y	0	0	y	y
z	0	0	z	z

(ii) Let N ={0,i,a,b,c,d,e,f,g} and let + be defined as in Table 7 and the ternary operation [ ] be defined as [f g h] = f ∘ g ∘ h where ∘ is defined as in Table 8. Then N is an RTNR. If Γ = $\mathbb{Z}_3$ = {0, 1, 2} then Γ is a right ternary N-group (using Table 9) and is monogenic by γ =1 w.r. to i, d, f, g. Also N is monogenic by γ =2 w.r.to i, d, f, g but N is *not* monogenic. It is also noted that N is strongly monogenic as $[Nx\gamma]_\Gamma$ = {0 } or Γ for all x∈N , γ∈ Γ.

Table 7

+	0	i	a	b	c	d	e	f	g
0	0	i	a	b	c	d	e	f	g
i	i	f	c	d	g	e	b	0	a
a	a	c	b	0	d	i	f	g	e
b	b	d	0	a	i	c	g	e	f
c	c	g	d	i	e	f	0	a	b
d	d	e	i	c	f	g	a	b	0
e	e	b	f	g	0	a	c	d	i
f	f	0	g	e	a	b	d	i	c
g	g	a	e	f	b	0	i	c	d

Table 8

$N \times N \times \Gamma$	0	1	2
0	0	0	0
i	0	1	2
a	0	0	1
b	0	0	2
c	0	1	0
d	0	1	1
e	0	2	0
f	0	2	1
g	0	2	2

Table 9

∘	0	i	a	b	c	d	e	f	g
∘	0	i	a	b	c	d	e	f	g
0	0	0	0	0	0	0	0	0	0
i	0	i	a	b	c	d	e	f	g
a	0	a	0	a	0	0	c	c	d
b	0	b	0	b	0	0	e	e	g
c	0	c	a	0	c	d	0	a	0
d	0	d	a	a	c	d	c	d	d
e	0	e	b	0	e	g	0	b	0
f	0	f	b	a	e	g	c	i	d
g	0	g	b	b	e	g	e	g	g

(iii) Let N = {0, x, y, z}. Define + as in Table 10 and the ternary operation [ ] on N by [xyz] = (x. y).z for every x, y, z in N where. is defined as in Table 11. Then (N, +, [ ]) is a right ternary near-ring. Let Γ = N. Then Γ is a right ternary N-group and is monogenic by γ for all $\gamma\in \Gamma$ as $[Nx\gamma] = \Gamma$ for all $x\in N$ and $\gamma\in \Gamma$.

Table 10

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table 11

.	0	x	y	z
0	0	0	0	0
x	x	x	x	x
y	y	y	y	y
z	z	z	z	z

Theorem 4.7 If N is a commutative RTNR and $_N\Gamma$ is faithful then N is embeddable in M (Γ).

Proof: Since $_N\Gamma$ is faithful, (o : Γ) = {0}. Define $h : N \to M(\Gamma)$ by $h(n) = f_n$, $\forall\ n \in N$ where $f_n : \Gamma\to\Gamma$ is defined by $f_n(\gamma) = [nx\gamma]_\Gamma\ \forall\ \gamma\in \Gamma,\ x \in N$. Obviously h is well defined. Consider ker h = $\{n \in N \mid h(n) = 0\} = \{n \in N \mid f_n = 0\} = \{n \in N \mid f_n(\gamma) = o,\ \forall\ \gamma\in \Gamma\} = \{n \in N \mid [n\ u\ \gamma]_\Gamma = o,\ \forall\ u \in N, \gamma\in \Gamma\} = (o : \Gamma) = \{0\}$, as $_N\Gamma$ is faithful.Thus h is 1-1. Obviously $h(n + n') = h(n) + h(n')$.

Consider $\forall\ x \in N$, $(f_n \circ f_{n'} \circ f_{n''})(\gamma) = f_n(f_{n'}(f_{n''}(\gamma))) = f_n(f_{n'}[n''\ x\ \gamma]_\Gamma) = f_n(n'\ u\ [n''\ x\ \gamma]_\Gamma) = [n\ v\ n'\ u\ [n''\ x\ \gamma]_\Gamma]_\Gamma = [[n\ v\ n'\ u\ n'']\ x\ \gamma]_\Gamma = [[n\ n'\ n'']\ [v\ u\ x]\ \gamma]_\Gamma = [[n\ n'\ n'']\ y\ \gamma]_\Gamma = f_{[n\ n'\ n'']}(\gamma) \Rightarrow f_n \circ f_{n'} \circ f_{n''} = f_{[n\ n'\ n'']}$. i.e., $h([n\ n'\ n'']) = h(n) \circ h(n') \circ h(n'')$ proving that h is an RTNR homomorphism. Hence N is embeddable in M (Γ).

Remark 4.8 In the above theorem the commutativity of N is necessary. For, consider N in Example4.2 (i) then N is a non- commutative RTNR. Let Γ={0,x,y}.Then (0:Γ) ={0} and if γ = y and u = 0 then $f_{[ycx]}(y) = 0$. Also $(f_y \circ f_c \circ f_x)$ (y) = [y m c n x l y]=[y x c x x x y]= y and hence $(f_y \circ f_c \circ f_x) \neq f_{[ycx]}$.

Theorem 4.9 Let a right ternary N-group Γ be faithful. Then the following statements are equivalent.

- $\Omega= \{o\}$
- $N_c = \{0\}$
- $N = N_o$

Proof: (i) $\Rightarrow$ (ii): By Proposition 3.4 (iv) $[N_c\ x\ \gamma]_\Gamma = [N_c\ 0\ o]_\Gamma\ \forall\ x \in N, \gamma \in \Gamma$.

$\Rightarrow [n_c\ x\ \gamma]_\Gamma = [n_c\ 0\ o] = o$ (given) $= [0\ x\ \gamma]_\Gamma$ by Proposition 3.4 (i)

$\Rightarrow [n_c - 0\ x\ \gamma]_\Gamma = 0 \Rightarrow n_c - 0 \in (o : \Gamma) = \{0\} \Rightarrow n_c = 0 \Rightarrow N_c = \{0\}$

$\Rightarrow$ (iii): Follows from the decomposition of N as $N = N_o + N_c$ and by hypothesis.

$\Rightarrow$ (i) Let $N = N_0$. Then $N_c = \{0\} \Rightarrow \Omega =[N_c\ 0\ o]_\Gamma = [0\ 0\ o]_\Gamma = \{o\}$.

Theorem 4.10 If ${}_N\Gamma$ is monogenic by γ_0 w.r.to $x\in N$ and L is a left ideal of N then $[L\ x\ \gamma_0]_\Gamma$ is an ideal of ${}_N\Gamma$.

Proof: If Γ is monogenic by γ_0 then $[N\ x\ \gamma_0]_\Gamma = \Gamma \Rightarrow \gamma = [n\ x\ \gamma_0]_\Gamma\ \forall\ \gamma \in \Gamma$. It can be easily seen that $[L\ x\ \gamma_0]_\Gamma$ is a normal subgroup of ${}_N\Gamma$. Consider for $\ell\in L$, $[n'\ x\ (\gamma + [\ell\ x\ \gamma_0]_\Gamma]_\Gamma - [n'\ x\ \gamma]_\Gamma= [n'\ x\ [n\ x\ \gamma_0]_\Gamma + [\ell\ x\ \gamma_0]_\Gamma]_\Gamma - [n'\ x\ \gamma]_\Gamma= [n'\ x\ [[n+\ell]\ x\ \gamma_0]_\Gamma] - [n'\ x\ [n\ x\ \gamma_0]]= [n'\ x\ (n+\ell) - [n'\ x\ n]\ x\ \gamma_0]_\Gamma\in [L\ x\ \gamma_0]_\Gamma$ as L is a left ideal of N.

Proposition 4.11 If e is a left unital element of N and ${}_N\Gamma$ faithful then e is a bi-unital element of N.

Proof: Let ${}_N\Gamma$ be faithful**.** Consider $[n - [n\ e\ e]\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma - [[n\ e\ e]\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma - [n\ [eex]\ \gamma]_\Gamma= [n\ x\ \gamma]_\Gamma - [n\ x\ \gamma]_\Gamma = o \Rightarrow [n - [n\ e\ e\ ]] \in (o : \Gamma) = \{0\} \Rightarrow n = [n\ e\ e]$.Thus e is a bi-unital element of N.

Proposition 4.12 If ${}_N\Gamma$ is monogenic by γ_0 then

- If e is a left unital element of N then $\forall\ \gamma \in \Gamma\ [e\ e\ \gamma]_\Gamma = \gamma$.
- If Γ is N_0-simple (Γ is considered as an N_0-group) then either $[N\ x\ \gamma]_\Gamma = \{o\}$ or Γfor each $x\in N$ and $\gamma\in\Gamma$.
- ${}_N\Gamma \cong_N N/(o: \gamma_0)$
- ${}_N\Gamma$ is simple iff $(o: \gamma_0)$ is a maximal left ideal of N or = N.

Proof: Let N be monogenic by γ_0. Then $[Nx\gamma_0]_\Gamma =\Gamma$ for each $x\in N \Rightarrow [nx\gamma_0]_\Gamma = \gamma\ \forall\ \gamma \in \Gamma$.

- Let e be a left unital element of N.Consider $[e\ e\ \gamma]_\Gamma = [e\ e\ [n\ x\ \gamma_0]_\Gamma]_\Gamma = [[e\ e\ n]\ x\ \gamma_0]_\Gamma = [n\ x\ \gamma_0]_\Gamma = \gamma_0$.
- Let Γ be N_0-simple.Now $[N\ x\ \gamma]_\Gamma = \{\ o\ \}$or $[N\ x\ \gamma]_\Gamma = \Gamma$, since for each $x\in N$, $[N\ x\ \gamma]_\Gamma$ is an N_0-subgroup of Γ.
- Define $h: N \to \Gamma$ by $h(n) = [n\ x\ \gamma_0]_\Gamma$ for each $x\in N$. Then $h(N) = \Gamma$ as N is monogenic by γ_0.Hence h is onto. Consider $h(n + n') = h(n) + h(n')$ and $h([n\ n'\ n'']) = [n\ n'\ [n''\ x\ \gamma_0]_\Gamma] = [n\ n'\ h(n'')]_\Gamma$. Thus h is an N- homomorphism and $\ker h = \{n \in N \mid [n\ x\ \gamma_0] = o\} = (o : \gamma_0)$.Hence $N/(o : \gamma_0) \cong_N {}_N\Gamma$.
- If ${}_N\Gamma$ is simple then $\{o\}$ and Γ are the only ideals of Γ. Now $(o: \gamma_0) = \{0\}$ or $(o: \gamma_0) =N$. since $(o: \gamma_0)$ is an ideal of N. If $(o: \gamma_0) = N$ then using (c), $(o: \gamma_0)$ is a maximal left ideal. If $(o: \gamma_0) = \{0\}$ then ${}_N\Gamma =N$. Conversely if $(o: \gamma_0)$ is a maximal left ideal of N then $N/ (o: \gamma_0)$ is simple. Hence by (c), ${}_N\Gamma$ is simple.

Proposition 4.13 If Δ is an N-subgroup of a right ternary N-group Γ and E is an ideal of Γ then Δ + E is an N-subgroup of Γ.

Proof: Since E is a normal subgroup of Γ and Δ is a subgroup, Δ + E is a subgroup of Γ. Let $\delta \in \Delta$, $\eta \in$ E. Consider for x,y $\in$ N , $[x\ y\ \delta + \eta]_\Gamma = [x\ y\ \delta]_\Gamma - [x\ y\ \delta]_\Gamma + [x\ y\ \delta + \eta]_\Gamma = [x\ y\ \delta]_\Gamma - ([x\ y\ \delta + \eta]_\Gamma - [x\ y\ \delta]_\Gamma) \in \Delta + E$. Hence Δ + E is an N-subgroup of Γ.

Remark 4.14 If Δ_1 and Δ_2 are N-subgroups then $\Delta_1 + \Delta_2$ need not be an N-subgroup. e.g Let N be as in Example 4.2 (i). Let $\Delta_1 = \{0,a\}$ and $\Delta_2 = \{0,b\}$.Then they are N-subgroups of Γ= N but $\Delta_1 + \Delta_2 = \{0,a,b,y\}$ is not a subgroup as y + y = x is *not* in $\Delta_1 + \Delta_2$.

Definition 4.15 A right ternary N-group Γ is called an *IFP right ternary N-group* if $[n\ x\ \gamma]_\Gamma = 0 \Rightarrow [n\ u\ x\ v\ \gamma]_\Gamma = 0$ for all u, v $\in$ N.

Remark 4.16 Every IFP-RTNR is an IFP right ternary N- group.

Definition 4.17 An element γ in a right ternary N-group Γ is called a *torsion free element* if for n $\in$ N, $[n\ x\ \gamma]_\Gamma = 0$ $\Rightarrow$ n = 0 for every x $\in$ N.

Example 4.18 (i) Let N = {0, x, y, z} and + be defined as in Table 12 and the ternary operation [ ] be defined as [a b c] = (a.b).c where. is defined as in Table 13.

Table 12

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table 13

.	0	x	y	z
0	0	0	0	0
x	0	0	x	x
y	0	0	z	y
z	0	0	y	z

Let Γ = N. Then x is a torsion free element in N for [n u x] = 0 $\forall$ u $\in$ N $\Rightarrow$ n = 0. Similarly y, z are also torsion free elements.

(ii) Let N = {0, x, y, z} and + be defined as in Table 14 and the ternary operation [ ] be defined as [a b c] = (a.b).c where is defined as in Table 15. Let Γ = N. Then x is not a torsion free element as [n u x] = 0 $\forall$ u $\in$ N holds even if n $\neq$ 0.

Table 14

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table 15

.	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
z	0	x	y	z

Lemma 4.19 If Γ is an IFP right ternary N-group with a torsion free element then N is an IFP-RTNR.

Proof: Let Γ be an IFP N-group and $\gamma \in \Gamma$ be a torsion free element. Let [x y z] =0for all x, y, z $\in$ N. Then by Proposition 3.4 (i), $\forall$ u $\in$ N, $[[x\ y\ z]\ u\ \gamma]_\Gamma = 0 \Rightarrow [x\ y\ [z\ u\ \gamma]_\Gamma]_\Gamma = 0 \Rightarrow [x\ m\ y\ n\ [z\ u\ \gamma]_\Gamma]_\Gamma = 0$ $\forall$ m,n $\in$ N $\Rightarrow$ $[[x\ m\ y\ n\ z]\ u\ \gamma]_\Gamma = 0 \Rightarrow [x\ m\ y\ n\ z] = 0$ as γ is a torsion free element. Hence N is an IFP-RTNR.

Corollary 4.20 If Γ is a faithful IFP right ternary N-group then N is an IFP-RTNR.

Proof: Let Γ be a faithful IFP right ternary N-group. Then $(o : \Gamma) = \{0\}$. This implies that for all $x \in N$ $[n\ x\ \gamma]_\Gamma = o$ with $n = 0$. Thus every element $\gamma \in \Gamma$ is torsion free and hence by the above lemma N is an IFP-RTNR.

5. TYPES OF N-GROUPS

In this section the three types of N-groups are defined and the relationships among them are established.

Definition 5.1 A monogenic N-group Γ with $\Gamma \neq \{o\}$ is said to be of *type* 0 if Γ is simple, *type* 1 if Γ is simple and strongly monogenic and *type* 2 if Γ is N_0-simple.

Proposition 5.2 Let Γ be an N-group. Then the following assertions hold.

- ${}_N\Gamma$ is of type 2 $\Rightarrow$ ${}_N\Gamma$is of type 1 $\Rightarrow$ ${}_N\Gamma$is of type 0.
- If ${}_N\Gamma$ is of type 1 or 2 then $\Omega = \{o\}$ or $\Omega = \Gamma$

Proof: (i) Let ${}_N\Gamma$ be of type 2. Then Γ is monogenic by γ(say) $\Rightarrow [N\ x\ \gamma]_\Gamma = \Gamma$ with $\Gamma \neq \{o\}$ and Γ is N_0-simple. If Δ is any proper ideal of Γ then as Γ is an N_0–group ,Δ will be a proper N_0–subgroup of Γ which is not possible as Γ is N_0-simple. Now as $N = N_0$, $\Omega =\{o\}$ and Γ are the only ideals of Γ and hence Γ is simple. Moreover as Γ is N_0-simple by Proposition 4.12(b) $\forall\ \gamma \in \Gamma$, $x \in N$ either $[N\ x\ \gamma]_\Gamma = \{o\}$ or Γ but $[N\ x\ \gamma]_\Gamma \neq \{o\}$ This implies that ${}_N\Gamma$ is strongly monogenic. Thus Γ is of type1. Now let ${}_N\Gamma$ be of type 1 then Γ is simple and hence is of type 0. (ii) Let ${}_N\Gamma$ be of type 1. Then $\{o\}$ and Γ are the only ideals of Γ and ${}_N\Gamma$ is strongly monogenic, i.e., $\forall\ \gamma \in \Gamma$, $x \in N$, $[N\ x\ \gamma]_\Gamma = \{o\}$ or $[N\ x\ \gamma]_\Gamma = \Gamma$ $\Rightarrow [N\ 0\ o]_\Gamma = \{o\}$ or $[N\ 0\ o]_\Gamma = \Gamma$. i.e., $\Omega = \{o\}$ or $\Omega = \Gamma$.If ${}_N\Gamma$ is of type 2 then by (i) it is of type 1 and hence $\Omega = \{o\}$ or $\Omega = \Gamma$.

Lemma 5.3 Let Γ be a right ternary N-group and Δ be a subset of Γ. Then

- ${}_N\Gamma$ is faithful iff ${}_{N_0}\Gamma$ and ${}_{N_C}\Gamma$ are faithful.
- Δ is an ideal of ${}_N\Gamma$ iff Δ is an ideal of ${}_{N_0}\Gamma$
- Δ is an N-subgroup of ${}_N\Gamma$ iff Δ is an N_0-subgroup of ${}_{N_0}\Gamma$

Proof

- Let $(o : \Gamma) = \{0\}$. Then $\{x \in N \mid [x\ n\ \gamma]_\Gamma = o\ \forall\ n \in N \text{ and } \gamma \in \Gamma\} = \{0\}$. In particular, $\{x \in N_0 \mid [x\ n\ \gamma]_\Gamma = o\ \forall\ n \in N, \gamma \in \Gamma\} = \{0\} \rightarrow (1)$and $\{x \in N_c \mid [x\ n\ \gamma]_\Gamma = o\ \forall\ n \in N \text{ and } \gamma \in \Gamma\} = \{0\} \rightarrow (2)$

  Conversely let (1) and (2) be true.Consider $x \in N$. Then $x = n_0 + n_c$. Now $[x\ n\ \gamma]_\Gamma = [n_0\ n\ \gamma]_\Gamma + [n_c\ n\ \gamma]_\Gamma = o + o = o$ for all $n \in N$ and $\gamma \in \Gamma$. Moreover $[n_0\ n\ \gamma]_\Gamma = o \Rightarrow n_0 = 0$ and $[n_c\ n\ \gamma]_\Gamma = o \Rightarrow n_c = 0$. Since $x = n_0 + n_c = 0 + 0 = 0$. Thus $[x\ n\ \gamma]_\Gamma = o$ with $x = 0$implies that ${}_N\Gamma$ is faithful.

- Let Δ be an ideal of ${}_N\Gamma$. Then $[x\ y\ \gamma+\delta]_\Gamma - [x\ y\ \gamma]_\Gamma \in \Delta$ for all $\gamma \in \Gamma$, $\delta \in \Delta$ and $x,y \in N$ and hence $[x\ y\ \gamma+\delta]_\Gamma - [x\ y\ \gamma]_\Gamma \in \Delta$ for all $\gamma \in \Gamma$, $\delta \in \Delta$ and $x,y \in N_0$ also. Hence Δ is an ideal of ${}_{N_0}\Gamma$. Conversely let Δ be an ideal of ${}_{N_0}\Gamma$. Consider for all $x, y \in N$, $[x\ y\ \gamma+\delta]_\Gamma - [x\ y\ \gamma]_\Gamma = [n_0+n_c\ y\ (\gamma+\delta)]_\Gamma - [n_0+n_c\ y\ \gamma]_\Gamma = [n_0\ y\ (\gamma+\delta)]_\Gamma + [n_c\ y\ (\gamma+\delta)]_\Gamma$

$-[n_c\ y\ \gamma]_\Gamma$ -$[n_0\ y\ \gamma]_\Gamma$= $[n_0\ y\ (\gamma+\delta)]_\Gamma + [n_c\ 0\ o]_\Gamma - [n_c\ 0\ o]_\Gamma - [n_0\ y\ \gamma] = [n_0\ y\ (\gamma+\delta)]_\Gamma - [n_0\ y\ \gamma] \in \Delta$.

- Let Δ be an N-subgroup of ${}_N\Gamma$. Then $[N\ N\ \Delta] \subseteq \Delta$. This implies that Δ is an N_0-subgroup as $[N_0\ N_0\ \Delta]_\Gamma \subseteq [N\ N\ \Delta]_\Gamma \subseteq \Delta$. Conversely let Δ be an N_0-subgroup of Γ. It is noted that $\Omega = [N_c\ 0\ o]_\Gamma \subseteq [N\ N\ \Delta]_\Gamma \subseteq \Delta$.Consider for x, y$\in$N, $[x\ y\ \delta]_\Gamma = [n_0+n_c\ y\ \delta]_\Gamma = [n_0\ y\ \delta]_\Gamma + [n_c\ y\ \delta]_\Gamma = [n_0\ y\ \delta] + [n_c\ 0\ o]_\Gamma \in \Delta$ as $\Omega \subseteq \Delta$.Thus Δ is an N-subgroup Γ.

Corollary 5.4 Let Γ be an N-group. Then

- ${}_N\Gamma$ is simple if and only if ${}_{N_0}\Gamma$ is simple.
- ${}_{N_0}\Gamma$ is monogenic by γ then ${}_N\Gamma$ is monogenic by γ.
- If ${}_{N_0}\Gamma$ is strongly monogenic then either ${}_N\Gamma$ is strongly monogenic or $\{o\} \neq \Omega \neq \Gamma$.
- If Γ is N_0-simple then Γ is N-simple.

Proof: (i) Let ${}_N\Gamma$ be simple. Then $\{o\}$ and Γ are the only ideals of ${}_N\Gamma$. Now by Lemma 5.3 (ii) $\{o\}$ and Γ are the only ideals of ${}_{N_0}\Gamma$. Hence ${}_{N_0}\Gamma$ is simple. Converse follows from Lemma 5.3 (ii).

(ii) Let $[N_0\ x\ \gamma]_\Gamma = \Gamma$ for all $x \in N$. Then as $[N_0\ x\ \gamma] \subseteq [N\ x\ \gamma]$, $\Gamma \subseteq [N\ x\ \gamma]_\Gamma$. Thus $[N\ x\ \gamma]_\Gamma = \Gamma$ as $[N\ x\ \gamma]_\Gamma \subseteq \Gamma$proving that ${}_N\Gamma$ is monogenic by γ.

(iii) Let ${}_{N_0}\Gamma$ be strongly monogenic. Then $[N_0\ x\ \gamma]_\Gamma = \{\ o\ \}$ or $[N_0\ x\ \gamma] = \Gamma\ \forall\ x \in N$ and $\gamma \in \Gamma$.

Case (i): Let $[N_0\ x\ \gamma]_\Gamma = \Gamma$ for all $x \in N$ and $\gamma \in \Gamma$.Then $[N_0\ x\ \gamma]_\Gamma = \Gamma + \Omega$. Hence ${}_N\Gamma$ is strongly monogenic.

Case (ii): If $[N_0\ x\ \gamma] = \{\ o\ \}$ then $[N\ x\ \gamma]_\Gamma = [N_0\ x\ \gamma]_\Gamma + [N_c\ x\ \gamma]_\Gamma$ and hence $[N\ x\ \gamma]_\Gamma = [N_c\ x\ \gamma]_\Gamma \Rightarrow \Omega \neq \{\ o\ \}$. Also $\Omega \subseteq \Gamma$. If $\Omega = \Gamma$ then Γ is strongly monogenic. Thus either ${}_N\Gamma$ is strongly monogenic or $\Omega \neq \{o\}$ and $\Omega \neq \Gamma$.

(iv) Let Γ be N_0-simple.Then Ω and Γ are the only N_0-subgroups of Γ and by Lemma 5.3 (iii) , Ω and Γ are the only N-subgroups of Γ and hence Γ is N-simple.

Note 5.5 The converse of (ii) need not be true as can be seen from Example4.5 (iii).

Theorem 5.6 Let ${}_N\Gamma$ be a right ternary N-group and $\upsilon \in \{0, 1, 2\}$. Then

- ${}_N\Gamma$ is of type $\upsilon \Rightarrow {}_{N_0}\Gamma$ is of type υ or $[N_0\ x\ \gamma]_\Gamma = \{o\}$ for all $\gamma \in \Gamma$.
- ${}_{N_0}\Gamma$ is of type υ (for $\gamma = 1$ let in ${}_N\Gamma$, $\Omega = \{o\}$ or $\Omega = \Gamma$) then ${}_N\Gamma$ is of type υ

Proof: (i) If ${}_N\Gamma$ is of type 0 then ${}_N\Gamma$ is simple. By Corollary 5.4 (i) this implies that ${}_{N_0}\Gamma$ is simple. Also since ${}_N\Gamma$ is monogenic by γ, $[N\ x\ \gamma]_\Gamma = \Gamma$. Hence by Lemma 5.3 (iii) $[N\ x\ \gamma]_\Gamma$ is an N_0-subgroup. By Theorem 4.10, $[N_0\ x\ \gamma]_\Gamma$ is a left ideal of ${}_N\Gamma$ as N_0 is a left ideal of N. Thus $[N_0\ x\ \gamma]_\Gamma = \{o\}$ or $[N_0\ x\ \gamma]_\Gamma = \Gamma$ as ${}_N\Gamma$ is simple. Suppose $[N_0\ x\ \gamma]_\Gamma = \Gamma$ then ${}_{N_0}\Gamma$ is monogenic. Thus ${}_{N_0}\Gamma$ is of type 0.If $[N_0\ x\ \gamma]_\Gamma = \{o\}$ then(i) follows. If ${}_N\Gamma$ is of type 1 then ${}_N\Gamma$ is simple and strongly monogenic. By Proposition 5.2 (ii) $\Omega = \{o\}$ or $\Omega = \Gamma$. If $\Omega = \{o\}$ then for every $\gamma \in \Gamma$, Γ= $[N\ x\ \gamma]_\Gamma = [N_0\ x\ \gamma]_\Gamma + [N_c\ x\ \gamma]_\Gamma$ =

$[N_0 \text{ x } \gamma]_\Gamma + \Omega = [N_0 \text{ x } \gamma]_\Gamma$. This implies that ${}_{N_0}\Gamma$ is strongly monogenic. Since ${}_N\Gamma$ is simple by Corollary 5.4 (i) ${}_{N_0}\Gamma$ is simple. Thus ${}_{N_0}\Gamma$ is of type 1. Now if $\Omega = \Gamma$ then $[N_0 \text{ x } \gamma]_\Gamma = \{o\}$ Hence either ${}_{N_0}\Gamma$ either is of type 1 or $[N_0 \text{ x } \Gamma]_\Gamma = \{o\}$. If ${}_N\Gamma$ is of type 2 then ${}_N\Gamma$ is N_0-simple. i.e., $\{o\}$ and Γ are the only N_0-subgroups and hence ${}_{N_0}\Gamma$ is N_0-simple. Thus ${}_{N_0}\Gamma$ is of type 2.

- (Let ${}_{N_0}\Gamma$ be of type 0 then ${}_{N_0}\Gamma$ is simple and monogenic. Hence by Corollary 5.4 (i) and (ii), ${}_N\Gamma$ is simple and monogenic. Thus ${}_N\Gamma$ is of type 0. Let ${}_{N_0}\Gamma$ be of type 1. i.e., ${}_{N_0}\Gamma$ is simple and strongly monogenic. Since ${}_{N_0}\Gamma$ is simple $\Omega = \{o\}$ or $\Omega = \Gamma$. Hence by Corollary 5.4 (iii), ${}_N\Gamma$ is strongly monogenic. Also by Corollary 5.4 (i) ${}_N\Gamma$ is simple. Hence ${}_N\Gamma$ is of type 1. Let ${}_{N_0}\Gamma$ be of type 2. Then ${}_{N_0}\Gamma$ is N_0-simple. Hence by Corollary 5.4 (iv) ${}_N\Gamma$ is N_0-simple. Hence ${}_N\Gamma$ is of type 2.

6. CONCLUSIONS

In this paper right ternary N-groups were defined and some of the properties of substructures of an RTNR, via right ternary N- groups were obtained. The results on faithful right ternary N-groups and monogenic right ternary N-groups introduced in this generalised setting can further be investigated in a zero-symmetric RTNR with desecending chain condition on N- subgroups. A characterisation theorem for a simple monogenic right ternary N-group Γ and an embedding theorem for a commutative RTNR were given. The inter relationship between the different types of right ternary N-groups can further be explored. This theory can further be developed by introducing υ- modular left ideals and υ-primitive RTNR.

REFERENCES

1. Lister, W.G. (1971). Ternary Rings, Transactions of American Math. Soc., 154.
2. Pilz, G. (1983), Near rings, Mathematic studies 23, North Holland Publishing Company.
3. Sukhendu Kar. (2010), Ternary semiring An Introduction, VDM Verlag Dr. Muller, 2010.
4. ISBN-10:3639004019ISBN-13: 978- 3639004014.
5. Uma Maheswari, A., & Meera, C. (2014), An Insight into Right Ternary Near-rings, International Journal of Mathematical Sciences, ISSN: 2051-5995, 34(1).
6. Uma Maheswari, A., & Meera, C. (2013), Fuzzy Soft Prime Ideals over Right Ternary Near-ring, International Journal of Pure and Applied Mathematics, 85(3), 507–529, doi: 10.12732/ijpam.v85i3.7.
7. Uma Maheswari, A., & Meera, C. (2013), Quasi-ideals and Bi-ideals of a Right Ternary Near-ring, International Journal of Scientific and Engineering Research, 4(5).
8. Uma Maheswari, A., & Meera, C. (2012), Fuzzy soft right ternary near-rings, International Journal of Computer Applications, 57(6).